SELF-SIMILARITY OF NONSTATIONARY BOUNDARY-LAYER MOTIONS

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In this paper we show that problems concerning the development of a boundary layer on a semi-infinite plate when the outer flow speed is of the form $U = (1 + ct)^b a$, and on a cylinder when the outer flow speed has the forms $U = ct^\alpha x^m$ and $U = (1 + ct)^b a x^m$, are self-similar. We present the results of numerical calculations for various values of α, b, and m. We consider the problem of a stepwise nonstationary heating of a plate, impulsively set into motion in an incompressible fluid; we show that this problem is self-similar and obtain its solution numerically.

Problem Concerning the Motion of a Semi-Infinite Plate with Speed $U = (1 + ct)^b a$

In this case the system of differential equations for the nonstationary laminar boundary layer and the associated boundary conditions have the form

$$\begin{gathered}\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0;\\ \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=acb(1+ct)^{b-1}+\nu\frac{\partial^2 u}{\partial y^2};\\ u=U,\ v=0 \text{ for } y=0,\ t=0;\\ u=0,\ v=0,\ y=0,\ t>0,\ x>0;\\ u=U,\ y=\infty,\ t>0,\ x>0.\end{gathered} \tag{1}$$

As new independent variables and functions we take the following:

$$\xi=\frac{x}{(1+ct)^{1+b}};\quad \eta=\frac{y}{(1+ct)^{1/2}};\quad u=\Phi(1+ct)^b;\quad v=\frac{V}{(1+ct)^{1/2}}.$$

With these substitutions the problem may be written in the form

$$\begin{gathered}\frac{\partial\Phi}{\partial\xi}+\frac{\partial V}{\partial\eta}=0;\\ bc\Phi-\frac{1}{2}c\eta\frac{\partial\Phi}{\partial\eta}-c(1+b)\xi\frac{\partial\Phi}{\partial\xi}+\Phi\frac{\partial\Phi}{\partial\xi}+V\frac{\partial\Phi}{\partial\eta}=acb+\nu\frac{\partial^2\Phi}{\partial\eta^2};\\ \Phi(\eta,\xi)=0,\ V=0 \text{ for } \eta=0;\\ \Phi(\eta,\xi)=1 \text{ for } \eta=\infty.\end{gathered}$$

Problem Concerning Acceleration of a Cylinder ($U = ct^\alpha x^m$)

The system of boundary-layer equations (see [1]) has the form

$$\begin{gathered}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\alpha c x^m t^{\alpha-1}+c^2t^{2\alpha}mx^{2m-1}+\nu\frac{\partial^2 u}{\partial y^2};\\ \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.\end{gathered} \tag{2}$$

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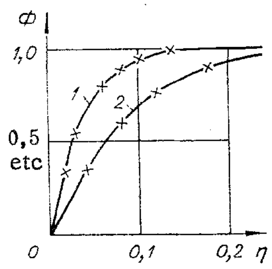

Fig. 1

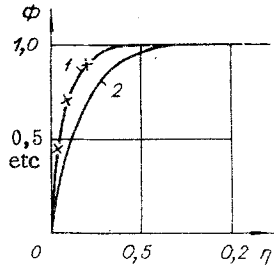

Fig. 2

The boundary conditions here are analogous to the conditions in problem (1). We introduce the new variables

$$\xi = x\left(t^{\frac{\alpha+1}{1-m}}\right)^{-1}, \qquad \eta = \frac{y}{t^{1/2}};$$

$$\Phi = u/cx^m t^{-\alpha}, \; V = v/cx^{m-1}t^{-(\alpha+1/2)}.$$

In terms of these new variables Eqs. (2) become

$$\alpha\Phi + \frac{(\alpha+1)}{m-1}\xi\frac{\partial\Phi}{\partial\xi} - \frac{1}{2}\eta\frac{\partial\Phi}{\partial\eta} + \Phi\left[cm\xi^{m-1}\Phi + c\xi^m\frac{\partial\Phi}{\partial\xi}\right] + c\xi^{m-1}V\frac{\partial\Phi}{\partial\eta} = \alpha + cm\xi^{m-1} + \nu\frac{\partial^2\Phi}{\partial\eta^2};$$

$$m\Phi + \xi\frac{\partial\Phi}{\partial\xi} + \frac{\partial V}{\partial\eta} = 0,$$

while the boundary conditions have the same form as in problem (1).

To solve this problem numerically we use the implicit numerical scheme (see [2])

$$\frac{\Phi_i^{j+1} - \Phi_i^j}{\Delta\xi} = \frac{1}{\left(\Phi_i^{j+1}c\xi_i^m + \frac{\alpha+1}{m-1}\xi_i\right)}\left[\frac{\Phi_{i+1}^{j+1} - 2\Phi_i^{j+1} + \Phi_{i-1}^{j+1}}{\Delta\eta^2} - \right.$$

$$\left. - \left(c\xi_i^{m-1}V_i^{j+1} - \frac{1}{2}\eta_i\right)\frac{\Phi_{i+1}^{j+1} - \Phi_i^{j+1}}{\Delta\eta} + cm\xi_i^{m-1}\left(1 - \Phi_i^{j+1}\right)^2 + \alpha\left(1 - \Phi_i^{j+1}\right)\right].$$

In calculating the flow field in the direction of the longitudinal coordinate η we use the driver method at each i-th point ξ_i. Figures 1 and 2 show the tangential velocity component for the ξ, η coordinate values used in the computations with the c value taken equal to 500 (in these figures curves 1 and 2 correspond to ξ values of 0.5 and 11, respectively). Figure 1 is for the case α = 0.5 and m = 0.5; Fig. 2 is for α = 0.5 and m = 1.5. The starred points are for α = 1.5 and m = 0.5 in Fig. 1; in Fig. 2 they are for α = 1.5 and m = 1.5. From the difference scheme used and from the graphs it is evident that the exponent α depends weakly on the dimensionless velocity profile, while m changes its form completely depending on the coordinates.

Problem Concerning the Motion of a Cylinder with the Speed $U = (1 + ct)^b \alpha x^m$

With the introduction of the variables

$$\xi = \frac{x}{(1+ct)^{\frac{1+b}{1-m}}}; \quad \eta = \frac{y}{(1+ct)^{1/2}}; \quad \Phi = \frac{u}{ax^m}(1+ct)^{-b};$$

$$V = \frac{v}{cx^{m-1}}(1+ct)^{-\left(\frac{1}{2}+b\right)}$$

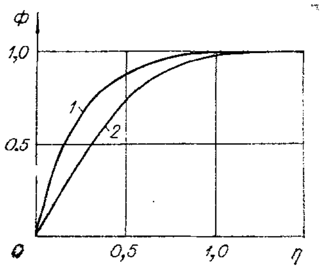

Fig. 3

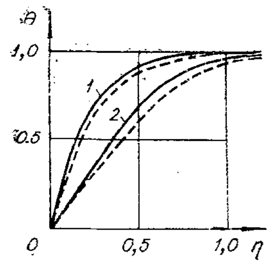

Fig. 4

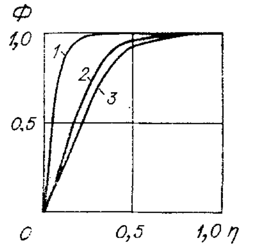

Fig. 5

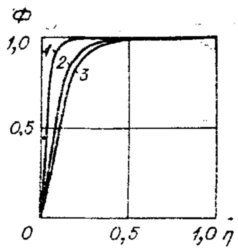

Fig. 6

this problem may be written in the form

$$bc\Phi - \frac{1}{2} c\eta \frac{\partial \Phi}{\partial \eta} - \frac{1+b}{1-m} c\xi \frac{\partial \Phi}{\partial \xi} + \Phi \left[a\xi^m \frac{\partial \Phi}{\partial \xi} + ma\xi^{m-1}\Phi \right] +$$
$$+ a\xi^{m-1} V \frac{\partial \Phi}{\partial \eta} = bc + am\xi^{m-1} + \nu \frac{\partial^2 \Phi}{\partial \eta^2}; \quad m\Phi + \xi \frac{\partial \Phi}{\partial \xi} + \frac{\partial V}{\partial \eta} = 0;$$
$$\Phi(\eta, \xi) = 0, \ V = 0 \ \text{for} \ \eta = 0;$$
$$\Phi(\eta, \xi) = 1 \ \text{for} \ \eta = \infty.$$

Since in the initial equations only the derivative with respect to t appears, we can state that if there is self-similarity for the functions $U = f(t)\varphi(x)$, there will also be self-similarity for the function $U = f(1 + ct)\varphi(x)$.

The numerical scheme used to obtain a solution is similar to that of the previous problem. An analysis of the results confirms the weak influence of the acceleration law which applies here; it also confirms the strong dependence of the dimensionless velocity profile on the form of the body (m).

The equations describing the problem concerning stepwise nonstationary heating of a flat plate, set impulsively into motion, are (see [1])

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0;$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}; \tag{3}$$
$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{\nu}{\sigma} \frac{\partial^2 T}{\partial y^2},$$

where σ is the Prandtl number, and

$$u = v = 0, \ T = T_e \ \text{for} \ t = 0;$$

$$u=v=0 \quad \text{for} \quad t>0,\ y=0;$$
$$u=U_\infty \quad \text{for} \quad t>0,\ x>0,\ y=\infty;$$
$$T=T_w \quad \text{for} \quad t>0,\ y=0,\ x>0;$$
$$T=T_e \quad \text{for} \quad t>0,\ y=\infty.$$

After substituting the variables

$$u=\Phi(\xi,\eta);\quad v=\frac{V}{t^{1/2}};\quad \xi=\frac{x}{t};\quad \eta=\frac{y}{t^{1/2}};\quad \theta=\frac{T-T_w}{T_e-T_w}$$

we can rewrite problem (3) in the form

$$\frac{\partial\Phi}{\partial\xi}+\frac{\partial V}{\partial\eta}=0;$$
$$-\xi\frac{\partial\Phi}{\partial\xi}-\frac{1}{2}\eta\frac{\partial\Phi}{\partial\eta}+\Phi\frac{\partial\Phi}{\partial\xi}+V\frac{\partial\Phi}{\partial\eta}=\nu\frac{\partial^2\Phi}{\partial\eta^2};$$
$$-\xi\frac{\partial\theta}{\partial\xi}-\frac{1}{2}\eta\frac{\partial\theta}{\partial\eta}+\Phi\frac{\partial\theta}{\partial\xi}+V\frac{\partial\theta}{\partial\eta}=\frac{\nu}{\sigma}\frac{\partial^2\theta}{\partial\eta^2};$$
$$\Phi=V=0,\quad \theta=0,\ \eta=0;$$
$$\Phi=U,\quad \theta=1,\ \eta=\infty.$$

The kinematic and temperature profiles for U_∞= 500 cm/sec are shown in Figs. 3 and 4 for σ = 0.7 (solid curve) and σ = 0.4 (dashed curve), respectively. Curves 1 and 2 correspond to ξ values of 0.25 and 17.75, respectively.

We employed the same methods to analyze the thermal boundary layer in the presence of a longitudinal pressure drop (impulsive Falkner–Skan motion) for small Mach numbers. The calculated results are shown in Figs. 5 and 6 for m = 0.5 and σ = 0.7, where the curves 1, 2, and 3 correspond to ξ values of 0.25, 17.75, and 37, respectively.

For the Falkner–Skan problem the thermal and kinematic profiles differ considerably from one another.

By using the self-similar variables we can carry out the calculations on a machine like the BESM-3M, a stationary solution being obtained after a computer time of some 20 to 30 min. One can also show, within the scope of the boundary-layer theory, that the problem concerning the impulsive motion of a plate in a compressible gas, as well as the problem concerning the formation of a boundary layer behind a transient shock wave on a thin semi-infinite plate, is self-similar.

LITERATURE CITED

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